TẠP CHÍ KHOA HỌC VÀ CÔNG NGHỆ ĐẠI HỌC DUY TÂN
DTU Journal of Science and Technology 04(65) (2024) 41-48
TRƯỜNG ĐẠI HỌC DUY TÂN
DUY TAN UNIVERSITY

Port-Hamiltonian modelling of two kinds of electrical circuits

Mô hình hóa Hamilton của hai kiểu mạch điện

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(Date of receiving article: 12/12/2023, date of completion of review: 23/03/2024, date of acceptance for posting: 13/05/2024)

Abstract

This work deals with port-Hamiltonian-based modelling of dynamical systems with application to electrical systems whose dynamics are *affine* in the control input. Two pH models of physical interest are proposed and compared, the first one is established with a series *RLC* circuit while the second one is obtained with a parallel *RLC* circuit. As the energy dissipation is due to the resistor, both models are associated with a quadratic Hamiltonian defining the total energy. Importantly, the circuit structure affects the pH formulation. Numerical simulations are carried out to illustrate the developed results.

Keywords: Electrical circuit; port-Hamiltonian representation; energy dissipation.

Tóm tắt

Bài báo xem xét vấn đề mô hình hóa Hamilton cổng của các hệ động lực với ứng dụng cho các hệ thống điện mà động lực là *affine* theo đầu vào điều khiển. Hai mô hình Hamilton có ý nghĩa vật lý được đề xuất và so sánh, biểu diễn thứ nhất được thiết lập với một mạch *RLC* mắc nối tiếp trong khi biểu diễn thứ hai nhận được với một mạch *RLC* mắc song song. Vì tiêu tán năng lượng gây ra do điện trở, cả hai mô hình được kết hợp với một hàm Hamilton toàn phương mô tả năng lượng tổng. Điểm thú vị là cấu trúc mạch ảnh hưởng kết quả thiết lập. Mô phỏng số được thực hiện để minh họa các kết quả.

Từ khóa: Mạch điện; biểu diễn Hamilton; tiêu tán năng lượng.

1. Introduction

This paper deals with dynamical systems [1, 2] whose dynamics are described by a set of Ordinary Differential Equations (ODEs) and *affine* in the input u as follows:

$$\frac{dx}{dt} = f(x) + g(x)u;\ x(t=0) = x_{init}, \qquad (1)$$

where $x = x(t)$ is the state vector contained in the operating region $D \subset \mathbb{R}^n$, $f(x) \in \mathbb{R}^n$ expresses the smooth function with respect to the

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vector x. The input-state map and the control input are respectively represented by $g(x) \in \mathbb{R}^{nxm}$ and $u \in \mathbb{R}^{m}$. Electrical, electromechanical or biochemical systems, etc. are typical examples of such systems [3-5].

From the energy-based point of view for modelling, writing the original dynamics (1) into the port-Hamiltonian (pH) representation is crucial to express the transformation of energy within the system [6, 7]. In other words, once a canonical form [8, 9], i.e. the pH representation of the dynamics (1), is somehow found, then a so-called energy balance equation (EBE) can be obtained. In turn, this equation allows expressing the transformation of energy, including the energy supply, storage and dissipation, etc. On the other hand, the resulting pH representation is well suited for passivity-based control [10, 11], control by interconnection [12-14], energy/power shaping control [6, 15] or setpoint tracking control [16]. Obtaining the pH representation of given dynamics is a key challenge in the structural modelling framework, and it is the main focus of this work.

This paper is organized as follows. Section 2 provides a brief overview of the pH representation of dynamical systems. Section 3 is devoted to the pH formulation of two dynamic electrical systems. Further discussions are also included. Section 4 ends the paper with some concluding remarks.

Notations: The following notations are considered throughout the paper:

- $\mathbb{R}$ is the set of real number.
- T is the matrix transpose operator.
- m and n $(m \leq n)$ are the positive integers.
- x_{init} is the initial value of the state vector.

2. An introductory overview of port-Hamiltonian systems

This section briefly recalls the fundamentals of port-Hamiltonian systems [8, 9] (see also [17]). Assume that the function $f(x)$ verifies the so-called separability condition [7, 18], that is, $f(x)$ can be decomposed and expressed as the product of some (interconnection and damping) structure matrices and the gradient of a potential function with respect to the state variables, i.e., the co-state variables:

$$f(x) = \left[\mathrm{J}(x) - \mathrm{R}(x)\right]\frac{\partial H(x)}{\partial x}, \tag{2}$$

where $\mathrm{J}(x)$ and $\mathrm{R}(x)$ are the $n \times n$ skew-symmetric interconnection matrix (i.e. $\mathrm{J}(x) = -\mathrm{J}(x)^{\mathrm{T}}$) and the $n \times n$ symmetric damping matrix (i.e. $\mathrm{R}(x) = \mathrm{R}(x)^{\mathrm{T}}$), respectively while $H(x): \mathbb{R}^{n} \rightarrow \mathbb{R}$ represents the Hamiltonian storage function of the system (possibly related to the total energy of the system). Furthermore, if the damping matrix $\mathrm{R}(x)$ is positive semi-definite, i.e.

$$\mathrm{R}(x) \geq 0, \tag{3}$$

then the original dynamics (1) is said to be a port-Hamiltonian (pH) representation with dissipation [8, 9]. Equation (1) is completed with the output and then rewritten as follows:

$$\begin{cases} \dfrac{dx}{dt} = \left[\mathrm{J}(x) - \mathrm{R}(x)\right]\dfrac{\partial H(x)}{\partial x} + g(x)u \\ y = g(x)^{\mathrm{T}}\dfrac{\partial H(x)}{\partial x} \end{cases} \tag{4}$$

where y is the output.

It can be clearly seen for the pH model defined by Eqs. (3) and (4) that the time derivative of the Hamiltonian storage function $H(x)$ satisfies the energy balance equation (EBE) [6].

$$\frac{dH(x)}{dt} = \underbrace{-\left[\frac{\partial H(x)}{\partial x}\right]^{\mathrm{T}} \mathrm{R}(x) \frac{\partial H(x)}{\partial x}}_{\text{dissipation}} + u^{\mathrm{T}} y. \quad (5)$$

It can be shown from Eq. (3) that the energy dissipation, defined by

$$d = -\left[\frac{\partial H(x)}{\partial x}\right]^{\mathrm{T}} \mathrm{R}(x) \frac{\partial H(x)}{\partial x} \le 0 \qquad (6)$$

is negative semi-definite. Hence, it represents a loss of energy due to resistive elements. The EBE (5) becomes:

$$\underbrace{\frac{dH(x)}{dt}}_{\text{stored power}} \le \underbrace{u^{\mathrm{T}} y}_{\text{supplied power}} . \qquad (7)$$

From a physical point of view, inequality (7) implies that the total amount of energy supplied from external source is always greater than the increase in the energy stored in the system. Hence, the pH system (4) is said to be *passive* with input u and output y corresponding to the Hamiltonian storage function [2] (we also refer the reader to [16] for further discussion).

In what follows, series and parallel circuits are used to illustrate and show the way to achieve a pH representation from given dynamics. For that purpose, the following lemma is adopted.

Lemma 1. Given a square matrix A. It follows that

$$A = \underbrace{\frac{A - A^{\mathrm{T}}}{2}}_{\text{skew-symmetric}} + \underbrace{\frac{A + A^{\mathrm{T}}}{2}}_{\text{symmetric}}.$$

3. Two case studies

3.1. Case study 1: A series circuit

3.1.1. Circuit description

We consider next a simple electrical system, which is the series circuit as sketched in Figure 1.

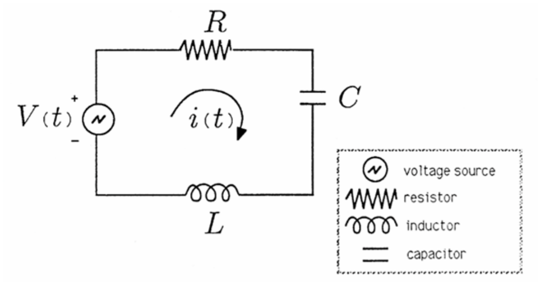


Figure 1. A series RLC circuit [19].

Before proceeding any further, we remind Kirchhoff's voltage law

$u_L + u_R + u_C = V$, (8)

and constitutive equations considered for three passive elements

$$\begin{cases} \text{the resistor } R: u_R = Ri_R, \\ \text{the inductor } L: \phi_L = Li_L \text{ and } u_L = \dfrac{d\phi_L}{dt}, \\ \text{the capacitor } C: i_C = \dfrac{dq_C}{dt} \text{ and } q_C = Cu_C, \end{cases} \qquad (9)$$

where q_C and ϕ_L are the charge stored in the capacitor C and the magnetic flux linkage through the inductor L, respectively; while i is the electric current passing through the circuit $(i = i_R = i_C = i_L)$ and u_L is the voltage of the inductor L (similarly for u_R and u_C).

3.1.2. Port-Hamiltonian formulation

Let $x := (q_C, \phi_L)^{\mathrm{T}}$ be the vector consisting of the charge q_C and the magnetic flux linkage ϕ_L. It can be shown from Eq. (9) that $\phi_L = L\dfrac{dq_C}{dt}$.

From Eqs. (8) and (9), one has [6, 20]:

$$\frac{dq_C}{dt} = \frac{1}{L}\phi_L, \qquad (10)$$

$$\frac{d\phi_L}{dt} = -\frac{1}{C}q_C - \frac{R}{L}\phi_L + V. \qquad (11)$$

Proposition 1 ([20]). Equations (10) and (11) constitute a pH representation described by (4) with $x := (q_C, \phi_L)^{\mathrm{T}}$ and

$$\mathrm{J}(x)=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \quad (12)$$

$$\mathrm{R}(x)=\begin{pmatrix}0 & 0\\ 0 & R\end{pmatrix}, \quad (13)$$

$$g(x)=\begin{pmatrix}0\\ 1\end{pmatrix}, \quad (14)$$

$$u=V, \quad (15)$$

$$y=\frac{1}{L}\phi_L. \quad (16)$$

Furthermore, the system is passive with the Hamiltonian defined by

$$H(x)=\frac{1}{2C}q_C^2+\frac{1}{2L}\phi_L^2. \quad (17)$$

Proof. It follows from Eqs. (1), (10) and (11) that $f(x)=\begin{pmatrix}\frac{1}{L}\phi_L\\ -\frac{1}{C}q_C-\frac{R}{L}\phi_L\end{pmatrix}$, which can be rewritten as $f(x)=\begin{pmatrix}0 & 1\\ -1 & -R\end{pmatrix}\begin{pmatrix}\frac{1}{C}q_C\\ \frac{1}{L}\phi_L\end{pmatrix}$. Let $A=\begin{pmatrix}0 & 1\\ -1 & -R\end{pmatrix}$, one may write $A=\mathrm{J}-\mathrm{R}$ using Lemma 1. This concludes the proof.

Remark 1. The Hamiltonian (17) is equal to the total energy of the system (i.e., it characterizes the amount of energy stored in capacitor and inductor). Hence it has the unit of energy [20].

Remark 2. From Eq. (6), it follows that

$$d=-R\left(\frac{1}{L}\phi_L\right)^2=-Ri^2\le 0, \quad (18)$$

which is precisely the power dissipated in the resistor.

3.2. Case study 2: A parallel circuit

3.2.1. Circuit description

Next, we consider a parallel circuit as sketched in Figure 2.

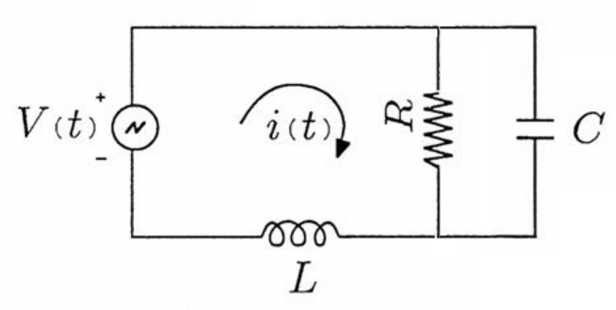


Figure 2. A parallel RLC circuit.

Kirchhoff's current and voltage laws in this case are

$$i=i_R+i_C, \quad (19)$$

$$u_L+u_C=V. \quad (20)$$

Note that $i=i_L$ and $u_R=u_C$.

3.2.2. Port-Hamiltonian formulation

Using Eqs. (9), (19) and (20), one obtains [6]:

$$\frac{dq_C}{dt}=-\frac{1}{RC}q_C+\frac{1}{L}\phi_L, \quad (21)$$

$$\frac{d\phi_L}{dt}=-\frac{1}{C}q_C+V. \quad (22)$$

Proposition 2. Equations (21) and (22) constitute a pH representation similar to that of Proposition 1 except

$$\mathrm{R}(x)=\begin{pmatrix}\frac{1}{R} & 0\\ 0 & 0\end{pmatrix}. \quad (23)$$

Proof. It follows from Eqs. (1), (21) and (22) that $f(x)=\begin{pmatrix}-\frac{1}{RC}q_C+\frac{1}{L}\phi_L\\ -\frac{1}{C}q_C\end{pmatrix}$, which can be rewritten as $f(x) = \begin{pmatrix} -\frac{1}{R} & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{C} q_C \\ \frac{1}{L}\phi_L \end{pmatrix}$.

Consider $A = \begin{pmatrix} -\frac{1}{R} & 1 \\ -1 & 0 \end{pmatrix}$, one may write $A = \mathrm{J} - \mathrm{R}$ using Lemma 1. This concludes the proof.

Remark 3. The Hamiltonian in this parallel *RLC* circuit is also equal to the total energy of the system. Hence it has the unit of energy.

Remark 4. From Eq. (6), it follows that

$$d = -\frac{1}{R}\left(\frac{1}{C} q_C\right)^2 = -\frac{1}{R} u_C^2 \le 0, \qquad (24)$$

which is also equal to the power dissipated in the resistor. This is because the damping matrix $\mathrm{R}(x)$ now has a similar structure (see Eqs. (13) and (23)) and $-\frac{1}{R}u_C^2 = -\frac{1}{R}u_R^2 = -Ri_R^2$.

Remark 4. For the case when the inductor is not ideal, i.e. it can be considered as a pure inductor connected in series with a resistor, the results in Propositions 1 and 2 remain valid with adequate modifications. For example, it can be shown using the same arguments that $\mathrm{R}(x) = \begin{pmatrix} 0 & 0 \\ 0 & R + R_L \end{pmatrix}$ for the series circuit, where R_L is the resistance of the inductor.

Table 1 summarizes the main features of the two proposed pH formulations.

Table 1. Features of the two pH formulations.

	The pH model with the series circuit	The pH model with the parallel circuit
x	$(q_C, \phi_L)^{\mathrm{T}}$	$(q_C, \phi_L)^{\mathrm{T}}$
$\mathrm{J}(x)$	is given by Eq. (12)	has the same form
$\mathrm{R}(x)$	**is given by Eq. (13)**	**given by Eq. (23)**
$\mathrm{g}(x)$	is given by Eq. (14)	has the same form
u	is given by Eq. (15)	has the same form
y	is given by Eq. (16)	has the same form
$H(x)$	is given by Eq. (17) (unit of energy)	has the same form

It is important to note that the energy dissipation in both formulations is strongly related to the value of the resistive element of the circuit, that is, the resistor. For the sake of illustration, Figure 3 shows the time evolution of state variables, while Figure 4 shows the dissipation of the two circuits where u is the Heaviside function (i.e. the unit step function) and the circuit elements are chosen as $R = 0.5(\Omega)$, $L = 6.25(\mathrm{H})$ and $C = 4(\mathrm{F})$ [19] (we refer the readers to Appendices A and B for the Simulink models). Unlike the the parallel circuit, the magnetic flux linkage through the inductor (or, equivalently, the current) of the series system is equal to 0 at permanent phase, there will be no dissipation in the resistor.

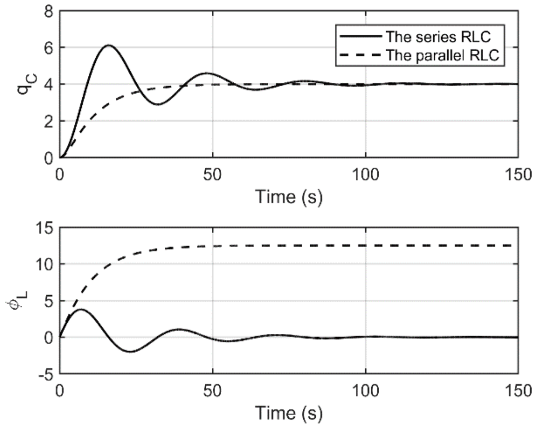


Figure 3. Time evolution of the states.

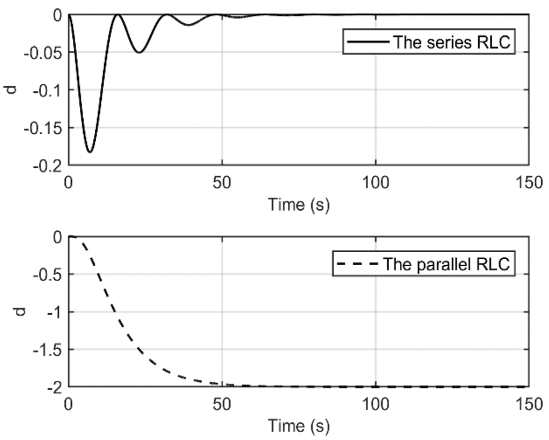


Figure 4. Dissipation of the two circuits.

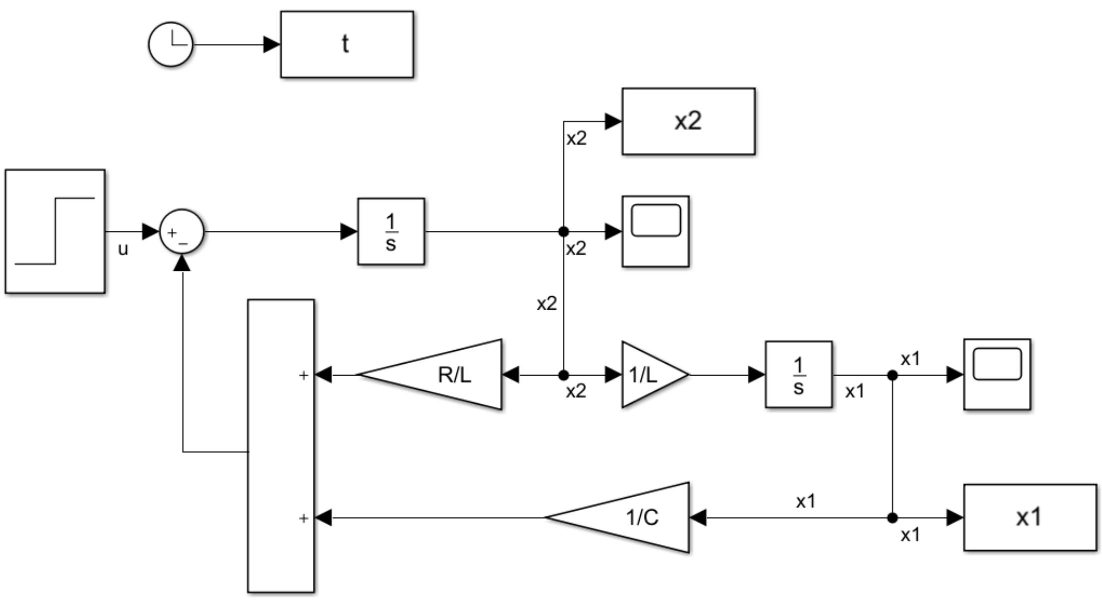
4. Conclusion

In this paper, the pH formulations of transient parallel and series *RLC* circuits are proposed and compared. The resulting Hamiltonian representations admit the quadratic Hamiltonian storage functions, which have the unit of energy while the energy dissipation is strongly related to the resistor. It remains now to adapt the setpoint tracking control theory [16] to stabilize the systems at a desired setpoint.

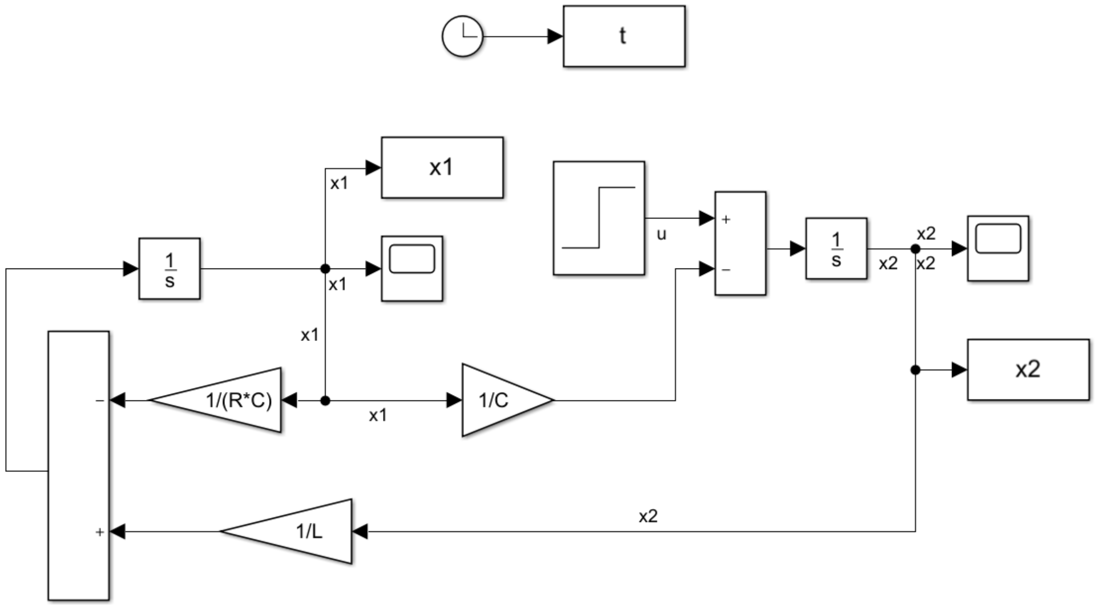
Acknowledgements

The authors express their gratitude to all the valuable support from Duy Tan University, who is going to celebrate its 30th anniversary of establishment (Nov. 11, 1994 - Nov. 11, 2024) towards "Integral, Sustainable and Stable Development".

Appendix A. The Simulink model of the series circuit



Appendix B. The Simulink model of the parallel circuit



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